1001. You are given that $-\frac{2}{3}$ is a root of f, where

$$f(x) = 6x^3 + 37x^2 - 41x - 42.$$

(a) Explain why $(3x + 2)$ must be a factor of $f(x)$.

(b) Factorise $f(x)$ fully.

1002. A die has been rolled. The score is X. Determine whether the fact "X is prime" increases, decreases or does not affect $\mathbb{P}(X \geq 3)$.

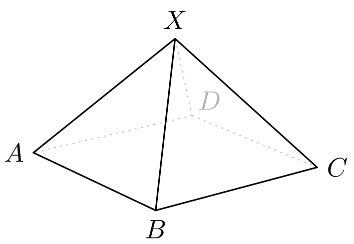
1003. Separate the variables in the following differential equation, writing it in the form $f(y)\frac{dy}{dx} = g(x)$ for some functions f and g:

$$x\frac{dy}{dx} + y^2 = 1.$$

1004. Prove that $\log_a b \times \log_b a \equiv 1$, for all $a, b > 0$.

1005. Determine whether the line $x = t, y = 2 - t$, for $t \in [0, 4]$ intersects the circle $x^2 + y^2 = 10$.

1006. The square-based pyramid shown below is formed of eight edges of unit length.



Determine angle BXD.

1007. Write down the angle between the vectors

(a) $\mathbf{i}$ and $\mathbf{j} + \mathbf{k}$,

(b) $\mathbf{j} + \mathbf{k}$ and $\mathbf{k}$.

1008. Show that, if the equation $y = 10x - 21 - x^2$ holds, then the inequality $y < x$ is always satisfied.

1009. State, with a reason, whether the following claims are true or false:

(a) "If triangles A and B have the same three side lengths, then they are congruent."

(b) "If parallelograms A and B have the same four side lengths, and also the same two diagonal lengths, then they are congruent."

1010. Prove that, if a quadratic graph $y = f(x)$ has a stationary point on the y axis, then it has no term in x.

1011. Is $\frac{dy}{dx} \cdot \frac{dx}{dy} \equiv 1$ true? Explain why.

1012. Find all (x, y) solution points which satisfy both $x^2 + y = 7$ and $(x + y)^2 = 25$.

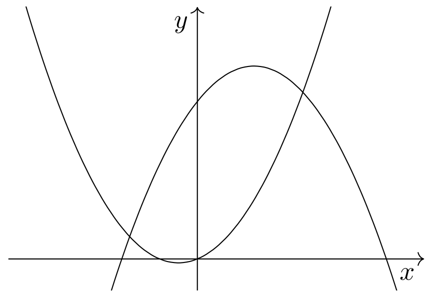
1013. On a single set of axes, sketch the following graphs, finding any points of intersection:

① $y = x^{\frac{1}{2}}$,

② $y = x^{\frac{1}{3}}$,

③ $y = x^{\frac{1}{4}}$.

1014. The graph shows two parabolae, $y = 2x^2 + x$ and $y = -2x^2 + 3x + 5$:



Describe fully the rotation which transforms the one into the other.

1015. Find simplified expressions for the sets

(a) $\{x \in \mathbb{R} : |x| < 2\} \cap [1, 3]$,

(b) $\{x \in \mathbb{R} : |x| > 2\} \cap [1, 3]$,

(c) $\{x \in \mathbb{R} : |x - 1| \leq 1\} \cap [1, 3]$.

1016. Explain the meanings of "census" and "sample".

1017. Find the sum of the first 100 multiples of 3.

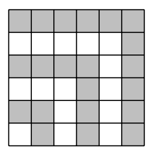
1018. True or false?

(a) $x \in A' \iff x \notin A$.

(b) $x \in A \iff x \notin A'$.

1019. Simplify $\lim_{p \to q} \frac{p^2 - q^2}{p - q}$.

1020. A *proof without words* is a proof which hopes to be self-evident visually. The following is one such:



Use the above to simplify $\sum_{r=1}^{n} (2r + 1)$.

1021. The graph $y = ax^2(x - b)$ has a local maximum at $(2, 4)$. Find the values of the constants a, b.

1022. Give the technical meaning of the following nouns used in mechanical modelling:

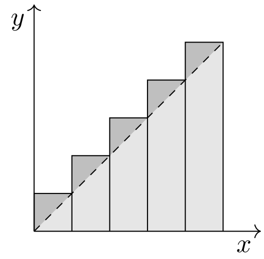
(a) "particle",
(b) "rod",
(c) "projectile".

1023. "The curves $x^2+y^2=2$ and $(x+2)^2+(y+2)^2=2$ are tangent to one another." True or false?

1024. By factorising, find all $x \in [0, 360°)$ such that

$$\cos^3 x = \cos x.$$

1025. The sum of the first n integers is represented in graphical form below, as an area.



By formulating expressions for the areas of the lighter and darker shaded regions, prove that

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1).$$

1026. The equations of parabolae P_1 and P_2 are given as $y=x^2$ and $y=-x^2+8x-15$.

(a) Show that $y=\frac{3}{2}-\frac{1}{2}x$ is normal to both.
(b) Hence, find the distance between P_1 and P_2.

1027. Explain how you know that a contact force exerted on a smooth sphere must have a line of action that passes through the sphere's centre.

1028. Prove, from first principles, that, if the derivative of f is f′, then, for any constant k,

$$\frac{d}{dx}\big(k\,\mathrm{f}(x)\big) = k\,\mathrm{f}'(x).$$

1029. Find the intersection of the diagonals of the convex quadrilateral with vertices at $(-4,8)$, $(4,9)$, $(12,0)$ and $(0,-1)$.

Note that the diagonals of a convex quadrilateral intersect inside the quadrilateral.

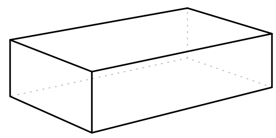
1030. State, with a reason, whether these hold:

(a) $a^2 \in \{0,1\} \iff a \in \{0,1\}$,
(b) $a^3 \in \{0,1\} \iff a \in \{0,1\}$,
(c) $a^4 \in \{0,1\} \iff a \in \{0,1\}$.

1031. At take-off, an aeroplane accelerates at $(a\mathbf{i}+b\mathbf{j})g$ ms^{-2}, where $\mathbf{i},\mathbf{j}$ are unit vectors in horizontal and vertical directions, and a,b are constants. Find the contact force on a passenger of mass m, giving your answer as a column vector with components in terms of a,b,m,g.

1032. Show that the y intercept of the line through the points (a,b) and $(2a,b+c)$ is independent of a.

1033. A cuboid has dimensions in the ratio $1:2:5$.



Its total surface area is 1152 cm^2. Find its volume.

1034. Events X and Y have probabilities $\mathbb{P}(X)={}^2/_3$ and $\mathbb{P}(Y)={}^1/_4$. Find all possible values of $\mathbb{P}(X'\cap Y')$.

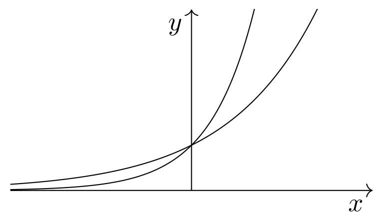
1035. Functions f and g are such that $x=a$ is a root of $\mathrm{f}(x)=0$, $x=b$ is a root of $\mathrm{g}(x)=0$, and $x=c$ is a root of $\mathrm{f}(x)=\mathrm{g}(x)$. State, with a reason, whether the following hold:

(a) If $a=b$, then $\mathrm{f}(a)=\mathrm{g}(a)$.
(b) If $a=b$, then $a=c$.

1036. Solve for a in the following equation:

$$\Big[x^2-2x\Big]_0^a = \Big[x^2+2x\Big]_0^2$$

1037. The graph shows $y=a^x$ and $y=b^x$, for constants $a,b>1$:



Show that $y=a^x$ is a stretch of $y=b^x$, and give the scale factor and direction of the enlargement.

1038. Explain, with reference to Newton's laws, which of the following objects would be worth grabbing if you are about to be blown (slowly and wearing a spacesuit) out of the airlock of a spacecraft:

- a vacuum cleaner,
- a machine-gun,
- a fan,
- a fire-extinguisher.

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK
WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

1039. Find the mean of the interior angles of a 16-gon, giving your answer in radians.

1040. Determine the value of $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$.

1041. A set of socks has seven pairs: two are labelled Monday, two Tuesday, etc. The socks are mixed up individually in a drawer, and I pick out two socks at random.

(a) Find the probability that they match.

(b) Given that they match, find the probability that they are the correct pair for that day.

1042. Using the binomial expansion, evaluate

$$\lim_{a \to 0} \frac{(1+a)^3 - 1}{(1+a)^4 - 1}.$$

1043. One of the following statements is true; the other is not. Identify and disprove the false statement.

(a) $\tan\theta = 1 \implies \theta = \frac{\pi}{4}$ rad,

(b) $\tan\theta = 1 \impliedby \theta = \frac{\pi}{4}$ rad.

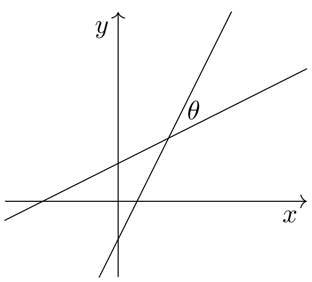
1044. A set of bivariate data (x_i, y_i) is being analysed. The mean $(\bar{x}, \bar{y})$ is found. A correlation coefficient r is then calculated, and a line of best fit passing through $(\bar{x}, \bar{y})$ is drawn on a scatter diagram.

Subsequently, it is discovered that $(\bar{x}, \bar{y})$ was itself mistakenly entered as a data point for calculation of r and the line of best fit. Describe the effect of removing it on

(a) r,

(b) the line of best fit.

1045. Two lines $y = m_1 x + c_1$ and $y = m_2 x + c_2$ are drawn, where $0 < m_1 < m_2$.



Find a formula for the acute angle θ.

1046. Prove that, according to the projectile model, the trajectory of a particle is symmetrical about its highest point.

1047. Two functions f and g are such that $f'(x) = g''(x)$ for all x. Either prove or disprove the following statement: "If $y = g(x)$ is stationary at $x = \alpha$, then $f(x) = 0$ has a root at $x = \alpha$."

1048. By quoting a standard derivative, show that

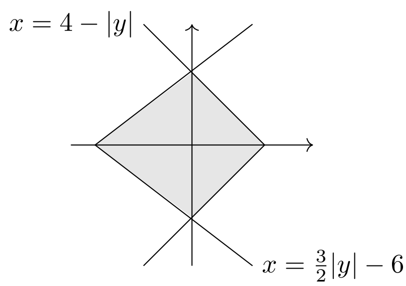
$$\int_0^{\frac{\pi}{4}} \sec^2 x \, dx = 1.$$

1049. A six-sided die and a twelve-sided die are rolled at the same time. Find the probability that the scores on the two dice are the same.

1050. Solve $\dfrac{1}{1 - \frac{1}{x} + \frac{1}{x^2}} = x$.

1051. The equation $\cos\theta = k$ has exactly one root in $[0, 360°)$. Determine the possible values of k.

1052. Two graphs are represented below:



Find the area of the shaded region.

1053. It is given that $x - 2y$ is constant. Find $\dfrac{dy}{dx}$.

1054. State, giving a reason, which of the implications $\implies$, $\impliedby$, $\iff$ (if any) links statements ① and ② concerning a real number x:

① $x \in A$,

② $x \in A \cap B$.

1055. A student has attempted to calculate the total area enclosed by the curves $y = x^3 - x$ and $y = 3x$ using the integral

$$I = \int_{-2}^{2} x^3 - 4x \, dx.$$

(a) Explain how the integrand has been obtained.

(b) Explain how the limits have been obtained.

(c) Explain what is wrong with the calculation.

(d) Show that the total area enclosed is 8.

1056. True or false?

(a) The sum of the first n integers is odd.

(b) The sum of the first n odd integers is odd.

(c) The sum of the first n even integers is even.

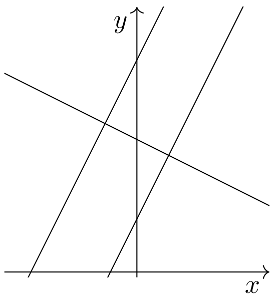
1057. Without expanding the brackets, solve

$$(3x-2)^2(x-1) + (3x-2)(x-1)^2 = 0.$$

1058. If $\dfrac{d}{dx}(x+y+1)=0$, find $\dfrac{dy}{dx}$.

1059. By considering the signs of the factors, solve the inequality $(x^2+1)(x-2)\geq 0$, giving your answer in set notation.

1060. A set of four lines forms a square. The first three are $y=2x+1$, $y=2x+4$ and $x+2y=5$, which are shown below.



Find the two possible equations of the last line.

1061. Verify that the parabola $y=x^2+2$ satisfies the differential equation $y\frac{dy}{dx}-4x=2x^3$.

1062. An irregular hexagon has perimeter 60 cm, and its shortest side has length l cm. Determine the set of possible values of l.

1063. Two functions f and g are such that, for all $x\in\mathbb{R}$,

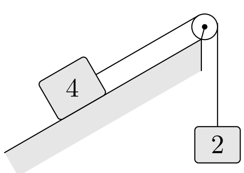
$$\tfrac{d}{dx}\big(\mathrm{f}(x)-\mathrm{g}(x)\big)=2.$$

Show that $\mathrm{f}(x)=\mathrm{g}(x)$ has exactly one root.

1064. If $z=a^x$, write $\left(\sqrt[3]{a}\right)^x$ in terms of z.

1065. Simplify $e^{3\ln a-\ln b}$.

1066. Two masses are connected by a light, inextensible string, which is passed over a smooth, light, fixed pulley as shown in the diagram. The 4 kg mass sits on a smooth slope, in equilibrium.



Determine the angle of inclination of the slope.

1067. Rectangle R has sides parallel to and three vertices on the coordinate axes. The curve $y=x^2$ passes through two of the vertices of R. Show that the curve divides R into two regions, whose areas are in the ratio $1:2$.

1068. Prove, by exhaustion or otherwise, that there are no integer solutions to $x^2+y^2=42$.

1069. Find the value(s) of p so that $\mathbf{r}=\frac{1}{7}\left(2\mathbf{i}+3\mathbf{j}+p\mathbf{k}\right)$ is a unit vector.

1070. The quadratic function $\mathrm{f}(x)=ax^2+8x-a$ has discriminant 100. Find all possible values of a.

1071. A hand of five cards is dealt from a standard deck. Find the probability that the hand is a flush, i.e. all the cards are the same suit.

1072. Solve $x-3x^{\frac{2}{3}}+2x^{\frac{1}{3}}=0$.

1073. State, with a reason, whether the graph $y=|x|$ intersects the following curves:

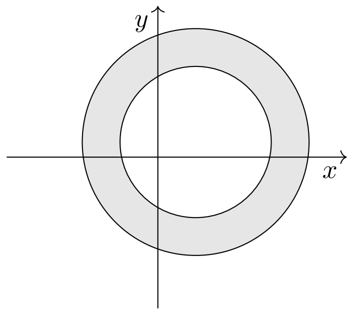
(a) $y=1-x^2$,

(b) $y=1+x^2$.

1074. An iteration is defined as $B_{n+1}=a(B_n+1)$, where a is a constant. You are given that $B_1=3$ and $B_3=18$. Find all possible values of a.

1075. The diagram shows an annulus defined by

$$a\leq (x-p)^2+(y-q)^2\leq b.$$



Find the area (shaded) of the annulus.

1076. Prove that, whatever the values of the constants a,b,c,d, the following function is not well-defined over $\mathbb{R}$:

$$\mathrm{f}(x)=\frac{(x^2-a^2-1)(x^2+b^2-1)}{(x^2-c^2-1)(x^2+d^2-1)}.$$

1077. State that the following holds, or explain why not: "A resultant moment of zero is necessary, but not sufficient, for an object to be in equilibrium."

1078. You are given that the line $x+3y=k$ is a normal to the curve $y=x^3$ at point P.

(a) Determine the possible coordinates of P.

(b) Find all possible values of k.

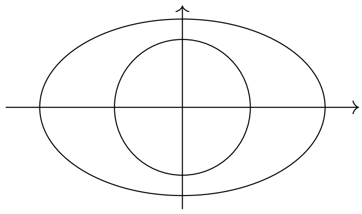
1079. By attempting the factorisation explicitly, prove that (x^2+1) is not a factor of $4x^5+x+1$.

1080. A trapeze artist of mass 55 kg swings across a stage, holding onto a light, rigid bar. At the lowest point of her swing, she experiences an acceleration of 6 ms^{-2} upwards. Find the force exerted on the bar by the trapeze artist at this instant.

1081. Solve the following simultaneous equations:

$$16x^2y^2 - 8xy + 1 = 0,$$
$$4x + 4y = 5.$$

1082. An ellipse may be viewed as a transformed circle.



By considering scale factors parallel to the x and y axes, write down the area of the ellipse given by the parametric equations $x = a\cos\theta$, $y = b\sin\theta$.

1083. The function f is linear, such that

$$\int_a^b \mathrm{f}(x)\,dx = 0.$$

Find $\mathrm{f}\left(\frac{1}{2}(a+b)\right)$.

1084. Prove that the area of a rhombus is $A = \frac{1}{2}pq$, where p and q are the lengths of its diagonals.

1085. The equation of a straight line, gradient m, passing through the point (a, b) is

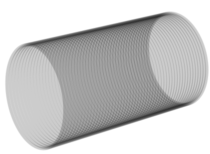
$$\frac{y-b}{x-a} = m.$$

Sketch the following graphs:

(a) $\dfrac{y-b}{(x-a)^2} = m,$

(b) $\dfrac{(y-b)^2}{x-a} = m.$

1086. In an industrial process, a carbon nanotube in the shape of a cylinder (with no ends) grows from an initial circular seed.



The radius is fixed, and the rate of change of length is constant. Explain whether the rate of change of surface area is constant, linear, quadratic, or none of the above.

1087. Determine which of the points $(4, 2)$ and $(3, 3)$ is closer to the circle $x^2 + y^2 = 1$.

1088. We define a logarithm by $a^x = b \iff \log_a b = x$. Prove, using this definition, that

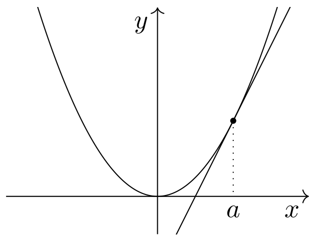
(a) $\log_a 1 = 0$,
(b) $\log_a a = 1$,
(c) $\log_a \sqrt{a} = \frac{1}{2}$.

1089. State, with a reason, which of the following shapes, in which diameter is defined vertex-to-vertex, has the larger area:

① a regular $2n$-gon of diameter d,
② a regular $(2n+2)$-gon of diameter d.

1090. Given $Z \sim N(0, 1)$, find $\mathbb{P}(Z^2 > Z)$.

1091. The parabola $y = x^2$ has a tangent T drawn to it at a point with x coordinate $a \neq 0$.



Show that T crosses the x axis at $x = a/2$.

1092. A sequence is defined by

$$u_{n+1} = u_n + n, \quad u_1 = 1.$$

(a) Give the first five terms.
(b) By considering the differences, find an ordinal nth term formula for the sequence.

1093. Find the radius of the smallest circle that could contain a $(3, 4, 5)$ triangle.

1094. Solve the simultaneous equations

$$2\sqrt{x} + 3y^2 = 18,$$
$$3\sqrt{x} - 2y^2 = 1.$$

1095. A cube of mass m is held in equilibrium against a rough vertical wall by a horizontal force. This force has magnitude $\frac{1}{2}mg$. Find the least possible value of μ, the coefficient of friction between the wall and the cube.

1096. The line $4x + y = 8$ can be expressed in the form $x = 1 - 2t$, $y = a + bt$. Find the constants a and b.

1097. Provide a counterexample to the following:

$$p = q \implies \frac{p-q}{r-s} = 0, \text{ for all } p, q, r, s \in \mathbb{R}.$$

1098. The quartic $y = x^4 - 8x^2 + 16$ has two double roots.

(a) Factorise the quartic fully, and hence find the double roots.

(b) Show explicitly, by finding the derivative, that the gradient is zero at the double roots.

(c) Hence, sketch the curve.

1099. Integers $a < b < c$ are such that the ratio $c - b : a - c$ is $2 : 3$. Prove that, if a is a multiple of 5, then so is b.

1100. Prove that, for $x, y > 0$,

$$\log_x y \equiv \frac{1}{\log_y x}.$$

——— End of 11th Hundred ———